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CUMULANT-BASED IDENTIFICATION OF NOISY CLOSED LOOP SYSTEMS

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SUMMARY

Conventional parameter estimation approaches fail to identify linear systems operating in closed loop when both input and output measurements are contaminated by additive noise of unknown (cross-)spectral characteristics. However, even in the absence of measurement noise, parameter estimation is involved owing to the additive system noise entering the loop. The present work introduces a novel criterion which is theoretically insensitive to a class of disturbances and yields the same parameter estimates that one obtains using mean squared error (MSE) minimization in the absence of noise. A strongly convergent sample-based approximation of the proposed criterion is introduced for consistent parameter estimation in practice. It is also shown that in the common case of ARMA modelling the resulting parameter estimates coincide with those obtained from a set of linear equations which can be solved using a time-recursive algorithm. Simulation results are presented to verify the performance of the proposed schemes in low-signal-to-noise-ratio environments.

KEY WORDS closed loop systems; system identification; parameter estimation; recursive estimation; statistics (cumulants)

1. INTRODUCTION

The operation of systems under feedback control is either a necessary engineering solution in order to stabilize the overall plant or in other cases a modelling scheme for more accurate description of the system dynamics. The first category includes industrial plants for the production of paper, cement and glass as well as navigation systems for ships and aircraft. The second category refers to physical, biological and economic systems.

In the present study we adopt the closed loop model referring indistinguishably to all the above systems and develop a parametric identification method for determining the open loop system even when disturbances of a certain class enter the loop and corrupt the input/output data. We examine the closed loop system depicted in Figure 1 assuming that the processes $w(t)$ and $z(t)$ are measurable and that $H_s(z)$ can be parametrized by a vector θ whose value is to be estimated.

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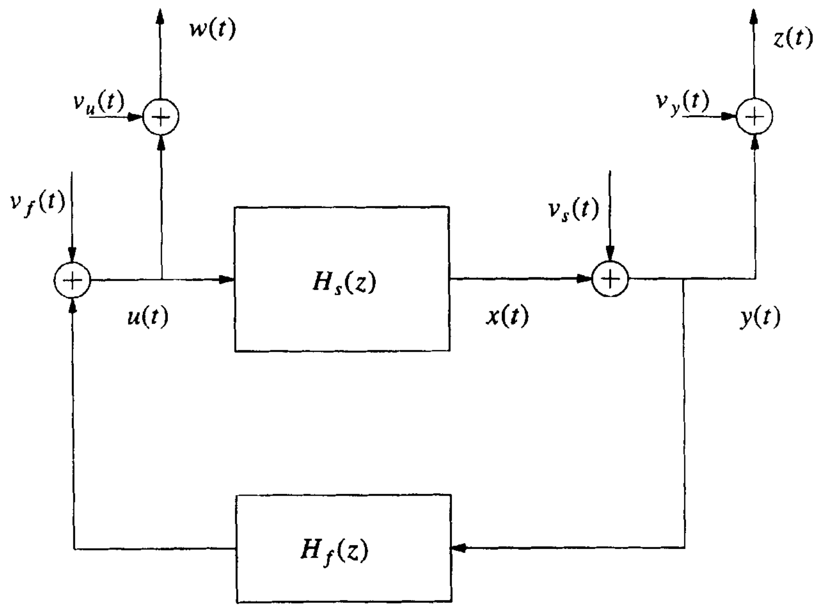

Figure 1.

An excellent review of existing identification methods that apply to closed loop systems can be found in Reference 1 (see also Reference 2). The observed processes in Reference 1 are assumed noise-free, i.e. $w(t) = u(t)$ and $z(t) = y(t)$. Although this restriction simplifies the problem significantly, it is known that

(i) (non-)parametric power spectral techniques do not provide unbiased estimates of $H_s(z)$
(ii) parametric techniques that employ minimization of prediction errors allow consistent estimates of the underlying parameters but require non-linear optimization.

In Reference 3 a non-parametric closed loop identification approach relied upon higher-than-second-order spectra in order to provide unbiased and consistent estimates of the open loop transfer function $H_s(z)$. Open loop identification with noisy input/output data (also known as errors-in-variables modelling (Reference 4, p. 203)) is a special case of the closed loop scenario and algorithms relying on higher-order statistics (HOS) have been reported in References 5–10. Akaike[11] was the first to report on the use of HOS for identification of closed loop systems.

The parametric and criterion-based method derived herein exploits the theoretical insensitivity of higher-than-second-order cumulants to certain kinds of additive noise in order to circumvent the effects of $v_s(t)$, $v_u(t)$, and $v_y(t)$ for the model of Figure 1. We essentially extend the input/output technique reported in References 6–8 to closed loop environments. The generalized prediction error (GPE) method that we propose turns out to be computationally attractive when $H_s(z)$ is modelled as a rational transfer function, since it reduces to solving a set of linear equations. In addition, a time-recursive RLS-type algorithm is delineated for solving the aforementioned equations.

Strong consistency of the parameter estimates is also presented. The identifiability and persistence-of-excitation conditions are no different from those encountered in the existing analysis (see e.g. References 1 and 8) and we will not repeat them here.

The rest of the paper is organized as follows. In Section 2 we describe the exact structure of the overall system and set the assumptions on the nature of the signals involved. Section 3 deals with a new noise-insensitive loss function $\bar{V}_\theta$, which replaces in our method the conventional mean squared error V_θ. A sample-based counterpart $\bar{V}_\theta^{(N)}$ of $\bar{V}_\theta$ is also introduced and its convergence to $\bar{V}_\theta$ w.p.1 and uniformly in θ as the number of samples $N \to \infty$ is proved. Consistency of the parameter estimator $\theta_N \triangleq \arg \text{extremum } \bar{V}_\theta^{(N)}$ is also established. In Section 4, the interesting case of ARMA modelling is examined and an appropriate recursive, pseudoadaptive algorithm for computing θ_N is outlined. Section 5 contains simulation results verifying the theoretical assessments of the previous section.

2. PROBLEM STATEMENT

In this section we define the structure of the system that we intend to identify and state the assumptions on the system and the signals involved. Figure 1 depicts the entire configuration. Denoting by q^{-1} the unit-delay operator, the above closed loop operation is summarized by the equations

$$y(t) = H_s(q)u(t) + v_s(t) \tag{1a}$$

$$u(t) = H_f(q)y(t) + v_f(t) \tag{1b}$$

$$w(t) = u(t) + v_u(t), \qquad z(t) = y(t) + v_y(t) \tag{2}$$

where all involved signals are scalar and all filters are single-input/single-output (SISO). Combining (1a) and (1b), we obtain direct expressions for $u(t)$ and $y(t)$ provided that

$$F(q) \triangleq [1 - H_s(q)H_f(q)]^{-1}$$

is stable; specifically, it follows readily that

$$u(t) = \frac{1}{1 - H_s(q)H_f(q)}\, v_f(t) + \frac{H_f(q)}{1 - H_s(q)H_f(q)}\, v_s(t) = u_f(t) + u_s(t) \tag{3a}$$

$$y(t) = \frac{H_s(q)}{1 - H_s(q)H_f(q)}\, v_f(t) + \frac{1}{1 - H_s(q)H_f(q)}\, v_s(t) = y_f(t) + y_s(t) \tag{3b}$$

Here $u_f(t)$, $y_f(t)$ and $u_s(t)$, $y_s(t)$ are obviously the components of $u(t)$ and $y(t)$ that depend on $v_f(t)$ and $v_s(t)$ respectively. The 'f-dependent' components of $u(t)$ and $y(t)$ form an input/output pair

$$y_f(t) = H_s(q)u_f(t) \tag{4a}$$

with

$$u_f(t) = F(q)v_f(t) \tag{4b}$$

These last two relations will be exploited later in order to reduce the closed loop identification to an input/output identification task (see Section 3).

We proceed to set working assumptions on the signals and systems.

(A1) $H_s(z)$ is stable, causal without direct term, i.e. $H_s(z^{-1} = 0) = 0$.

(A2) $F(z) = [1 - H_s(z)H_f(z)]^{-1}$ is stable and $F(0) \triangleq F(z = e^{j\omega})|_{\omega=0} \neq 0$; the latter will be necessary in the next section in order to establish an important relation between $\bar{V}_\theta$ and V_θ (see (7) and also (8)).

(A3) $v_f(t) = L(q)v_{f0}(t)$, where $L(q)$ is a stable filter with $L(0) \neq 0$ and $v_{f0}(t)$ is a zero-mean, i.i.d. process with non-vanishing third-order cumulants, finite moments up to order six and $\gamma_2 = E\{v_{f0}^2(t)\}$, $\gamma_3 = E\{v_{f0}^3(t)\{$. We refer the reader to Reference 1, Chap. 2 for definitions and properties of cumulants.

(A4) $v_s(t)$ is a zero-mean, symmetrically distributed, possibly coloured process independent of $v_{f0}(t)$.

(A5) Measurement noises $v_u(t)$ and $v_y(t)$ are zero-mean, symmetrically distributed with unknown (cross-)power spectra independent of $v_{f0}(t)$.

Assumption (A1) and the stability of $F(z)$ in (A2) are standard (Reference 1, p. 383), while (A3)–(A5) define the class of disturbances that can be tolerated when third-order cumulants are used. They are satisfied for example when the PDF of $v_{f0}(t)$ is skewed and $\{v_s(t), v_u(t), v_y(t)\}$ are Gaussian and independent of $v_{f0}(t)$. However, if higher-than-third-order cumulants are employed, the condition $F(0) \neq 0$ can be dropped and we then require $\{v_s(t), v_u(t), v_y(t)\}$ to have vanishing kth-order cumulant, while it suffices for $v_{f0}(t)$ to have non-vanishing cumulant of the same order $k < 3$.

3. THE CRITERION AND ITS SAMPLE ESTIMATE

In this section we propose *direct identification* of $H_s(z)$ using a novel generalized prediction error method. When using the conventional MSE criterion, one faces two major problems in the identification procedure.

(i) I/O noises $v_u(t)$ and $v_y(t)$ of unknown colour introduce unknown biases in the estimates of $H_s(z)$. Essentially this happens because the MSE is sensitive to any kind of additive disturbances.

(ii) Even when $v_u(t) = v_y(t) \equiv 0$, i.e. $u(t)$ and $y(t)$ are directly accessible, the presence of $v_s(t)$ makes it necessary to employ a non-linear and/or iterative minimization procedure (see Reference 4, Chap. 10).

The proposed loss function exploits the theoretical insensitivity of third-order cumulants to additive noise with symmetric PDFs (see e.g. References 9 and 12) in order to overcome (i) and (ii) above.

Definition 1

For the structure of Figure 1 we define

$$\bar{V}_\theta \triangleq \sum_{k=-\infty}^{\infty} E\{w(t+k)\varepsilon_{w,z,\theta}^2(t)\} \tag{5}$$

where

$$\varepsilon_{w,z,\theta}(t) \triangleq D_{1,\theta}(q)w(t) + D_{2,\theta}(q)z(t) \tag{6}$$

is a generalized prediction error expressed as a linear combination of lags of $w(t)$ and $z(t)$. The FIR filters $D_{1,\theta}(q)$ and $D_{2,\theta}(q)$ are parametrized by θ and determine the structure of the predictor. They are also in one-to-one correspondence with the adopted model $H_{s,\theta}(z)$ that is to be fitted to the true system $H_s(z)$ (see also Reference 4, p. 169 for details on the parametrization of a model set).

Criterion $\bar{V}_\theta$ of (5) will turn out to be theoretically insensitive to noise satisfying (A3)–(A5) and was originally introduced for input/output identification purposes in Reference 8 and earlier for output-only identification in Reference 7. In this paper we extend its use to closed loop system identification; we prove that its major properties remain unaffected by the existence of the feedback loop. The following theorem establishes the equivalence of $\bar{V}_\theta$ to the conventional MSE when the latter is computed in the absence of disturbances $v_u(t)$, $v_y(t)$ and $v_s(t)$.

Theorem 1

If assumptions (A1)–(A5) hold and the generalized prediction error filters are, uniformly in θ, bounded and absolutely summable, then

$$\bar{V}_\theta = \alpha V_\theta \tag{7}$$

where $V_\theta \triangleq E\{\varepsilon^2_{uf,yf,\theta}(t)\}$ and $\alpha \triangleq (\gamma_3/\gamma_2)F(0)L(0)$.

Proof. Starting with (2) and (6), we first observe that

$$E\{w(t+k)\varepsilon^2_{w,z,\theta}(t)\} = E\{u(t+k)\varepsilon^2_{u,y,\theta}(t)\}$$

since $\{v_u(t), v_y(t), v_s(t)\}$ are independent of $\{v_{f0}(t)\}$ and, owing to (A4) and (A5), they have vanishing (cross-)third-order cumulants. Similarly, using the decomposition of (3a,b) and the fact that $v_s(t)$ is independent of $v_f(t)$ and has zero third-order cumulant, we find

$$E\{u(t+k)\varepsilon^2_{u,y,\theta}(t)\} = E\{u_f(t+k)\varepsilon^2_{uf,yf,\theta}(t)\}$$

where $\varepsilon_{uf,yf,\theta}(t) \triangleq D_{1,\theta}(q)u_f(t) + D_{2,\theta}(q)_f(t)$. Exploiting next the input/output relation (4a), the fact that $u_f(t) = F(q)L(q)v_{f0}(t)$ and using Theorem 3.1 of Reference 8, we conclude that

$$\sum_{k=-\infty}^{\infty} E\{w(t+k)\varepsilon^2_{w,z,\theta}(t)\} = \frac{\gamma_3}{\gamma_2} F(0)L(0)E\{\varepsilon^2_{uf,yf,\theta}(t)\}$$

which completes the proof.

According to (7) and since (A2) and (A3) guarantee $\alpha \neq 0$, it follows that the conventional MSE V_θ can be obtained from $\bar{V}_\theta$ within a scalar ambiguity. Although the sign of α is not known, Theorem 1 implies that extremization of $\bar{V}_\theta$ is equivalent to minimization of V_θ w.r.t. θ:

$$\theta_o \triangleq \arg\ \text{extremum}\ \bar{V}_\theta = \arg\min V_\theta \tag{8}$$

Since the latter involves only $u_f(t)$ and $y_f(t)$, which are related through the noise-free input/output relation (4a), estimates of θ obtained via (8) will be valid estimates of the true parameters of $H_s(z)$.

Following Reference 8, $\bar{V}_\theta$ of (5) can be generalized to cost functions $\bar{V}_{k,\theta}$ which are (modulated) projections of higher-than-third-order cumulants. Specifically, under (A1), (A2), $v_{f0}(t)$ being non-Gaussian with finite moments and $v_s(t)$, $v_u(t)$ and $v_y(t)$ Gaussian and independent of $v_{f0}(t)$, it follows from Lemma 4 of Appendix II that

$$\begin{aligned}\bar{V}_{k,\theta} &\triangleq \sum_{\tau_1}\cdots\sum_{\tau_{k-2}} \exp\left(-j\sum_{i=1}^{k-2}\omega_i\tau_i\right)\text{cum}\{w(t+\tau_1),\ldots,w(t+\tau_{k-2}),\varepsilon_{w,z,\theta}(t)\}\\ &= \left(\frac{\gamma_k}{\gamma_2}\right)\prod_{i=1}^{k-2}F(\omega_i)L(\omega_i)V_\theta \quad \text{for} \sum_{i=1}^{k-2}\omega_i = 0\end{aligned} \tag{9}$$

Cost function $\bar{V}_\theta$ is a special case of (9) with $k=3$. If k is even with $\omega_{2i+1} = \omega_0$ and $\omega_{2i} = -\omega_0$, we find: $\bar{V}_{k,\theta} = (\gamma_k/\gamma_2)|F(\omega_0)L(\omega_0)|^{k-2}V_\theta$. Since $v_{f0}(t)$ was assumed non-

Gaussian, $\gamma_k \neq 0$ for at least one $k \geq 3$. Also, $F(\omega_0)L(\omega_0) \neq 0$ for at least one $\omega_0 \in (-\pi, \pi]$. In practice $k=3$ or 4 will be adequate in most cases. For brevity we restrict ourselves to $k=3$.

So far we have shown that under (A1)–(A5) the loss function $\bar{V}_\theta$ is a noise-insensitive 'metric' for parameter identification of systems operating under feedback control. In practice, however, $\bar{V}_\theta$ cannot be computed since it involves expectations and infinite sums. In the sequel we define a sample-based approximation of $\bar{V}_\theta$, namely $\bar{V}_\theta^{(N)}$, and prove its strong convergence to $\bar{V}_\theta$ uniformly in θ as the length of the data record $N \to \infty$.

Definition 2

Let $\varepsilon_{w,z,\theta}(t)$ be defined as in (6). Let also $\lambda^*(k) \triangleq \lambda(k/M)$ be a discrete version of the continuous window $\lambda(t)$ defined on $[-1,1]$ which obeys the restrictions

$$\text{(R1)}\quad M = O(N^{1/(4+\delta)}), \quad \delta > 0 \tag{10a}$$

$$\text{(R2)}\quad \lambda(0) = 1, \qquad \lambda(-t) = \lambda(t) \tag{10b}$$

$$\text{(R3)}\quad \lim_{t\to 0} \frac{1-\lambda(t)}{t^2} = L < \infty \tag{10c}$$

$$\text{(R4)}\quad \sum_k \lambda^2\left(\frac{k}{M}\right) < \infty, \quad \forall N \tag{10d}$$

Then define

$$\bar{V}_\theta^{(N)} \triangleq \sum_{k=-N}^{N} \lambda^*(k)\left(\frac{1}{N}\sum_{t=1}^{N} w(t+k)\varepsilon_{w,z,\theta}^2(t)\right) \tag{11}$$

Comparing Definitions 1 and 2, two comments are in order: first, expectation in the definition of $\bar{V}_\theta$ has been replaced by time averaging; second, infinite summation over k has been replaced by a weighted sum over all available samples. It turns out that 'windowing' by $\lambda^*(k)$, constrained by (R1)–(R4), makes $\bar{V}_\theta^{(N)}$ a consistent estimator of $\bar{V}_\theta$.

The asymptotic behaviour of $\bar{V}_\theta^{(N)}$ is formally described by the following theorem.

Theorem 2

Let $\bar{V}_\theta$ be defined by (5) and $\bar{V}_\theta^{(N)}$ be as in (11). If (A1)–(A5) hold and

$$\tilde{V}_\theta^{(N)} \triangleq \sum_{k=-N}^{N} \lambda^*(k) E\{u_f(t+k)\varepsilon_{u_f,y_f,\theta}^2(t)\} \tag{12}$$

then

$$\text{(a)}\quad \sup_\theta |\bar{V}_\theta^{(N)} - \tilde{V}_\theta^{(N)}| \to 0 \quad \text{w.p.1} \quad \text{as } N \to \infty \tag{13a}$$

$$\text{(b)}\quad \sup_\theta |\tilde{V}_\theta^{(N)} - \bar{V}_\theta| \to 0 \quad \text{as } N \to \infty \tag{13b}$$

$$\text{(c)}\quad \sup_\theta |\tilde{V}_\theta^{(N)} - \bar{V}_\theta| \to 0 \quad \text{w.p.1} \quad \text{as } N \to \infty \tag{13c}$$

Proof. Before proceeding to the proof of (13), we quote a lemma that will be used in the proof.

Lemma 1

Let $x_\theta(t)$, $y_\theta(t)$ and $z_\theta(t)$ be linear processes such that

$$x_\theta(t) = \sum_k h_{x\theta}(t-k)e_x(k) \tag{14a}$$

$$y_\theta(t) = \sum_k h_{y\theta}(t-k)e_y(k) \tag{14b}$$

$$z_\theta(t) = \sum_k h_{z\theta}(t-k)e_z(k) \tag{14c}$$

and

(i) $$\sup_\theta \sum_t |h_{x\theta}(t)| \le C_x \tag{14d}$$

$$\sup_\theta \sum_t |h_{y\theta}(t)| \le C_y \tag{14e}$$

$$\sup_\theta \sum_t |h_{z\theta}(t)| \le C_z \tag{14f}$$

(ii) $e_x(t)$, $e_y(t)$ and $e_z(t)$ are i.i.d. with finite moments
(iii) $\lambda^*(k)$ obeys (R1)–(R4)
(iv) (a) $e_x(t)$ independent of $\{e_y(t),\ e_z(t)\}$ or (b) there exist t_1 and t_2 such that $e_x(t) = e_y(t-t_1) = e_z(t-t_2) = e(t)$.

Then

$$\sup_\theta \left| \frac{1}{N} \sum_{k=-N}^{N} \sum_{t=0}^{N} \lambda^*(k)[x_\theta(t+k)y_\theta(t)z_\theta(t) - E\{x_\theta(t+k)y_\theta(t)z_\theta(t)\}] \right| \to 0 \quad \text{w.p.1} \quad \text{as } N\to\infty \tag{15}$$

Proof. See Appendix I.

We now return now to the proof of Theorem 2.

(a) According to (2) and (3a, b), we have $w(t) = u_f(t) + u_s(t) + v_u(t)$ and $z(t) = y_f(t) + y_s(t) + v_y(t)$. Hence

$$\varepsilon_{w,z,\theta}(t) = \varepsilon_{u_f,y_f,\theta}(t) + D_{1,\theta}(q)[u_s(t) + v_u(t)] + D_{2,\theta}(q)[y_s(t) + v_y(t)] \tag{16}$$

Based on (16), we split $\bar{V}_\theta^{(N)}$ into two parts, namely one that involves $u_f(t)$ and $y_f(t)$ only and a second one that also includes the noise processes $v_u(t)$, $v_y(t)$ and $v_s(t)$:

$$\bar{V}_\theta^{(N)} = \frac{1}{N} \sum_{k=-N}^{N} \sum_{t=0}^{N} \lambda^*(k) u_f(t+k)\varepsilon^2_{u_f,y_f,\theta}(t) + \frac{1}{N} \sum_{k=-N}^{N} \sum_{t=0}^{N} \lambda^*(k) \quad \text{(triple cross-product terms)} \tag{17}$$

We examine the first part of the RHS of (17). In view of the definition of $\varepsilon_{u_f,y_f,\theta}(t)$, we have

$$u_f(t+k)\varepsilon^2_{u_f,y_f,\theta}(t) = u_f(t+k)\varepsilon^2_{1,\theta}(t) + u_f(t+k)\varepsilon^2_{2,\theta}(t) + 2u_f(t+k)\varepsilon_{1,\theta}(t)\varepsilon_{2,\theta}(t) \tag{18a}$$

where

$$u_f(t) = F(q)L(q)v_{f0}(t)$$
$$\varepsilon_{1,\theta}(t) \triangleq D_{1,\theta}(q)F(q)L(q)v_{f0}(t)$$
$$\varepsilon_{2,\theta}(t) \triangleq D_{2,\theta}(q)H_s(q)F(q)L(q)v_{f0}(t)$$

Clearly the above processes fulfil the conditions of Lemma 1. Hence for each of the three terms of (18a) and therefore, owing to the triangular inequality, for their sum the assertion of the lemma holds, i.e.

$$\sup_{\theta}\left|\frac{1}{N}\sum_{k=-N}^{N}\sum_{t=0}^{N}\lambda^*(k)[u(t+k)\varepsilon^2_{u_f,y_f,\theta}(t) - E\{u(t+k)\varepsilon^2_{u_f,y_f,\theta}(t)\}]\right| \to 0 \quad \text{w.p.1} \quad \text{as } N\to\infty \tag{18b}$$

On the other hand, all processes involved in the second part of (17) fulfil the requirements of Lemma 1 and, owing to (A4) and (A5), all triple products have expected values equal to zero. Hence, applying Lemma 1 again we conclude that the second term of (17) tends w.p.1 and uniformly in θ to zero as $N\to\infty$.

In view of (7) and (13b), this completes the proof of part (a) of the theorem.

(b) Part (b) can be established by following exactly the same arguments we used in Reference 7, Appendix C.

(c) Equation (13c) is a direct consequence of (13a) and (13b).

The uniform-in-θ convergence implied by Theorem 2 makes it possible to identify the parameters of $H_s(z)$ based on the noisy measurements $w(t)$ and $z(t)$. The following corollary to the theorem formalizes this claim.

Corollary

Under the assumptions of Theorem 2 and if

$$\theta_N \triangleq \arg\text{ extremum } \bar{V}_\theta^{(N)} \tag{19a}$$

$$\theta_o \triangleq \arg\min V_\theta \tag{19b}$$

then

$$\theta_N \to \theta_o \quad \text{w.p.1} \quad \text{as } N\to\infty \tag{19c}$$

The corollary suggests a non-linear method for estimating θ_o by extremizing $\bar{V}_\theta^{(N)}$. Convergence in (19c) should be interpreted in the sense of Reference 4, p. 215, which allows a set of extrema rather than a single global extremum. In addition, as we see next, the non-linear optimization can be reduced to solving linear equations when the assumed model is ARMA.

4. ARMA MODELLING

In many applications the transfer function $H_s(z)$ is approximated by an ARMA(P,Q) model. In this case the parameter vector θ includes the coefficients of the numerator and denominator polynomials and the generalized prediction error becomes

$$\varepsilon_{w,z,\theta}(t) = z(t) - \mathbf{x}^{\mathrm{T}}(t)\theta \tag{20}$$

where the augmented data vector $\mathbf{x}(t)$ is defined as

$$\mathbf{x}(t) \triangleq [z(t-1), z(t-2), \ldots, z(t-Q), w(t-1), w(t-2), \ldots, w(t-P)]^{\mathrm{T}} \tag{21}$$

Substituting (20) and (21) into (5), we obtain

$$\begin{aligned}\bar{V}_\theta &= \sum_{k=-\infty}^{\infty} E\{w(t+k)[z^2(t) + \theta^{\mathrm{T}}\mathbf{x}(t)\mathbf{x}^{\mathrm{T}}(t)\theta - 2z(t)\mathbf{x}^{\mathrm{T}}(t)\theta]\} \\ &= \sum_{K=-\infty}^{\infty} E\{w(t+k)z^2(t)\} + \theta^{\mathrm{T}}\left(\sum_{k=-\infty}^{\infty} E\{w(t+k)\mathbf{x}(t)\mathbf{x}^{\mathrm{T}}(t)\}\right)\theta \\ &\quad -2\left(\sum_{k=-\infty}^{\infty} E\{w(t+k)z(t)\mathbf{x}^{\mathrm{T}}(t)\}\right)\theta \end{aligned} \tag{22}$$

The extremizer θ_o is then the solution of the *normal* equations

$$\left(\sum_{k=-\infty}^{\infty} E\{w(t+k)\mathbf{x}(t)\mathbf{x}^{\mathrm{T}}(t)\}\right)\theta = \sum_{k=-\infty}^{\infty} E\{w(t+k)z(t)\mathbf{x}^{\mathrm{T}}(t)\} \tag{23}$$

Following similar steps, we conclude that the data-based estimate θ_N of θ is the solution of the *normal* equations

$$\Phi(N)\theta_N = \phi(N) \tag{24}$$

where

$$\Phi(N) \triangleq \sum_{t=0}^{N} s_w^N(t)\mathbf{x}(t)\mathbf{x}^{\mathrm{T}}(t) \tag{25a}$$

$$\phi(N) \triangleq \sum_{t=0}^{N} s_w^N(t)z(t)\mathbf{x}(t) \tag{25b}$$

with

$$s_w^N(t) \triangleq \sum_{k=-N}^{N} \lambda^*(k)w(t+k) \tag{25c}$$

The linear equations (24) accept a time-recursive solution similar to the recursive least squares (RLS) algorithm. The corresponding pseudoadaptive algorithm which is summarized in Table I is capable of tracking slow variations in $H_s(z)$ owing to the forgetting factor μ that

Table I. The proposed recursive algorithm

Step 1	$s_w(N) = \sum_{k=-M}^{M} \lambda^*(k)w(N+k)$
Step 2	$\mathbf{K}(N) = \dfrac{s_w(N)\mathbf{P}(N-1)\mathbf{x}(N)}{\mu + s_w(N)\mathbf{x}^{\mathrm{T}}(N)\mathbf{P}(N-1)\mathbf{x}(N)}$
Step 3	$\mathbf{a}(N) = \mathbf{z}(N) - \mathbf{x}^{\mathrm{T}}(N)\theta_{N-1}$
Step 4	$\theta_N = \theta_{N-1} + \mathbf{K}(N)\mathbf{a}(N)$
Step 5	$\mathbf{P}(N) = \dfrac{1}{\mu}[\mathbf{P}(N-1) - \mathbf{K}(N)\mathbf{x}^{\mathrm{T}}(N)\mathbf{P}(N-1)]$

Initialization: batch estimation of $\mathbf{P}(N_0) = \Phi(N_0)^{-1}$

has been included. Note also that computing $s_u^N(t)$ at every t requires nothing but a tap delay line.

V. SIMULATIONS

Simulated examples were used to verify the performance of the proposed identification scheme and to compare it with the performance of the conventional MSE-based method. A hundred Monte Carlo runs were performed in each experiment in order to approximate the mean value and standard deviation of the parameter estimates. In the following test cases the closed loop environment of Figure 1 was implemented using Matlab 4.0 on a SPARC station IPX.

Experiment 1

In the first experiment the direct branch system was chosen to be ARMA(2,1) with transfer function

$$H_s(z) = \frac{0{\cdot}5z^{-1}}{1 - 0{\cdot}4z^{-1} + 0{\cdot}5z^{-2}}$$

while the feedback transfer function was the AR(2) system

$$H_f(z) = \frac{1}{1 + 0{\cdot}3z^{-1} - 0{\cdot}2z^{-2}}$$

The input $v_f(t)$ was white, zero-mean, exponentially distributed with distribution parameter $\lambda = 1{\cdot}0$. The system noise $v_s(t)$ was generated by passing white, zero-mean Gaussian noise through an FIR filter with taps $[1, -2{\cdot}33, 0{\cdot}75, 0{\cdot}5, 0{\cdot}3, -1{\cdot}4]$, while the measurement disturbances $v_u(t)$ and $v_y(t)$ were zero-mean, normally distributed and mutually correlated through

$$v_y(t) = v_u(t) + 0{\cdot}2v_u(t-1) - 0.3v_u(t-2) + 0{\cdot}4v_u(t-3)$$

with noise $v_u(t)$ being white. In this experiment we set $SNR_s \triangleq 10\log_{10}[E\{v_f^2(t)\}/E\{v_s^2(t)\}]$ to 5 dB, $SNR_u \triangleq 10\log_{10}[E\{u^2(t)\}/E\{v_u^2(t)\}]$ to 5 dB and $SNR_y \triangleq 10\log_{10}[E\{y^2(t)\}/E\{v_y^2(t)\}]$ also to 5 dB. The length of the data records was $N = 4000$ points. The discrete window $\lambda^*(k)$ was a 25—point Hamming window. The first row of Table II contains the true ARMA parameters. The Monte Carlo mean ± standard deviation of the estimates obtained by solving

Table II. ARMA closed loop identification (Experiment 1)

Coefficient	b_0	b_1	a_1	a_2
True	0·000	0·500	−0·400	0·500
Proposed estimates	−0·014 ±0·096	0·513 ±0·079	−0·414 ±0·154	0·522 ±0·172
Conventional estimates	0·370 ±0·004	0·278 ±0·007	0·021 ±0·012	0·146 ±0·009

$N = 4000$, 100 MC runs, $SNR_u = 5$ dB, $SNR_y = 5$ dB, $SNR_s = 5$ dB.

the normal equations (24) and (25) are in the second row of the same table, while the estimates obtained by solving the autocorrelation-based normal equations are in the third row of the same table. Figures 2(a) and 2(b) depict the true (full curve) and estimated transfer functions of $H_s(z)$ (mean value only). In accordance with our theoretical assertions, the proposed estimates have converged to the true parameter values; a rather small bias is the result of using Monte Carlo mean instead of ideal expectations and also due to the use of a finite length window $\lambda^*(k)$ instead of one with increasing support $M(N) \to \infty$ (refer to restriction (R1), equation (10)).

Experiment 2

In the second experiment we used as direct branch system the ARMA(1, 1) transfer function

$$H_s(z) = \frac{0{\cdot}5z^{-1}}{1 + 0{\cdot}5z^{-1}}$$

while the remaining signals and systems involved were the same as in Experiment 1. We changed the SNR levels to $SNR_s = 5$ dB, $SNR_u = 3$ dB and $SNR_y = 3$ dB. The length of the data records was in this case $N = 2000$ points. Table III contains the corresponding Monte Carlo mean ± standard deviation estimates obtained by solving (24) and (25) in the case of the proposed ones and the conventional normal equations in the second and third rows respectively. Figures 3(a) and 3(b) depict the true and estimated (Monte Carlo mean) transfer functions of $H_s(z)$. The conventions regarding the curve-types are as in Experiment 1.

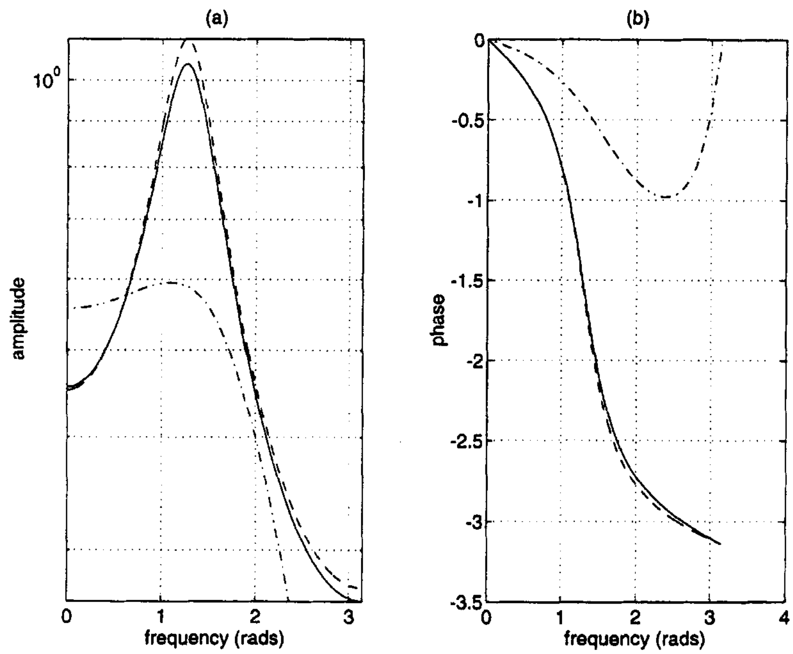

Figure 2. Experiment 1: (a) True and estimated (mean) transfer function amplitude; (b) true and estimated (mean) transfer function phase

Table III. ARMA closed loop identification (Experiment 2)

Coefficient	b_0	b_1	a_1
True	0·000	0·500	0·500
Proposed estimates	−0·014 ±0·052	0·511 ±0·044	0·505 ±0·111
Conventional estimates	0·290 ±0·010	0·424 ±0·013	−0·454 ±0·014

$N = 2000$, 100 MC runs, $SNR_u = 3$ dB, $SNR_y = 3$ dB, $SNR_s = 3$ dB.

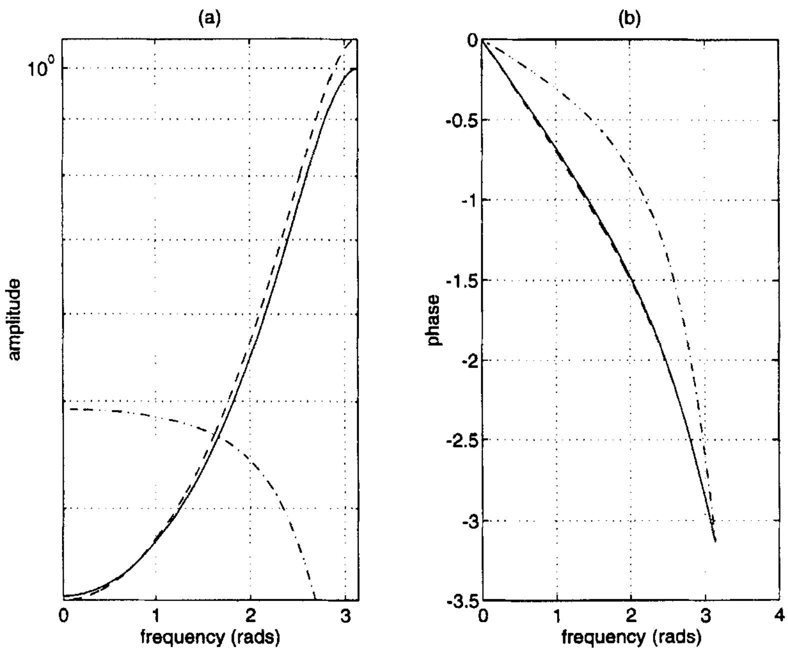

Figure 3. Experiment 2: (a) True and estimated (mean) transfer function amplitude; (b)true and estimated (mean) transfer function phase

6. CONCLUSIONS

Conventional identification techniques fail to provide unbiased estimates of systems operating in closed loop when the input and output measurements are contaminated by additive noises of unknown covariance. This is due to the well-known sensitivity of power spectra and second-order correlations to all kinds of additive noise. In addition, even in the absence of measurement noise identification of such systems requires iterative minimization and non-linear programming when system noise $v_s(t)$ enters the loop.

In this paper we introduced a parametric, time domain method for identification of LTI systems operating under feedback which is theoretically insensitive to a class of noises. The proposed identification approach is based on the extremization of a loss function $\bar{V}_\theta$ that is theoretically tolerant to all symmetrically distributed system and measurement disturbances $v_s(t)$, $v_u(t)$ and $v_y(t)$. The aforementioned insensitivity results from the dependence of V_θ purely on third-order cumulants of the input/output signals.

We showed that when both input and output records are contaminated by non-skewed noises, possibly coloured and even mutually correlated and also correlated to the system noise, the proposed loss function is equivalent to the conventional MSE when the latter is expressed in terms of the noise-free input and output of the system $H_s(z)$. Intuitively speaking, our kth-order (cross-)cumulant-based criterion separates disturbances from information signals on the basis of their distributional characteristics such as skewness ($k=3$) and kurtosis ($k=4$).

A sample estimate of the above criterion was established to be convergent strongly and uniformly in θ to the theoretical value of the criterion. As a corollary, practical and strongly consistent estimation of the parameters was guaranteed.

In particular, when the unknown transfer function is modelled as rational, the necessary extremization is attained by the solution of a system of linear equations. A time-recursive, pseudoadaptive algorithm similar to RLS was presented to implement the aforementioned solution.

Simulated experiments were used to verify the performance of the proposed technique in noisy closed loop environments.

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APPENDIX I: PROOF OF LEMMA 1

The proof of Lemma 1 is based upon two lemmas which deal with bounds and convergence of stochastic sums. The proofs of these lemmas can be found in Reference 7 (see also Reference 4).

Lemma 2

Let $\phi = \sum_{k=-\infty}^{\infty} a(k)z(k)$, where (i) $a(k)$ is a deterministic sequence satisfying $\sum_{k=-\infty}^{\infty} |a(k)| \leq C_a$ and (ii) $z(k)$ is a sequence of random variables such that $E\{z^2(k)\} \leq C_z$. Then $E\{\phi^2\} \leq C_z[\sum_{k=-\infty}^{\infty} |a(k)|]^2 \leq C_z C_a^2$.

Lemma 3

Let $\{X_t\}$ be a sequence of zero-mean random variables and define $Y_r^N \triangleq \sum_{t=r+1}^{N} X_t$, $0 \leq r \leq N$. If $E\{(Y_r^N)^2\} \leq CM^2(N)(N-r)$, where $M(N) \leq O(N^{l/(4+\delta)})$, $\delta > 0$, then the sequence $N^{-l}|Y_0^N| \to 0$ w.p.1 as $N \to \infty$.

We now proceed with the proof of Lemma 1. Define

$$
\begin{aligned}
D_\theta(r,N) &\triangleq \sum_{t=r+1}^{N} \left(\sum_{k=-N}^{N} \lambda^*(k)[x_\theta(t+k)y_\theta(t)z_\theta(t) - E\{x_\theta(t+k)y_\theta(t)z_\theta(t)\}] \right) \\
&= \sum_{k=-N}^{N} \lambda^*(k) \sum_i h_{x\theta}(i) \sum_j h_{y\theta}(j) \sum_l h_{z\theta}(l) \\
&\quad \times \sum_{t=r+1}^{N} [e_x(t+k-i)e_y(t-j)e_z(t-l) - E\{e_x(t+k-i)e_y(t-j)e_z(t-l)\}]
\end{aligned}
\tag{26}
$$

We intend to bound $E\{|D_\theta(r,N)|^2\}$ in order to use Lemma 2. To this end we first derive bounds for the individual deterministic and stochastic time series involved in (26). We first observe that since

$$\lim_{N\to\infty} M^{-1} \sum_{k=-N}^{N} |\lambda^*(k)| = \int_{-1}^{1} |\lambda(t)|\, dt$$

then for N large enough there exists a C_l such that

$$\sum_{k=-N}^{N} |\lambda^*(k)| \leqslant C_l M \tag{27}$$

On the other hand, we have bounds for the absolute sums of $h_{x\theta}(t)$, $h_{y\theta}(t)$ and $h_{z\theta}(t)$ owing to (14d–f). We next consider the stochastic term. If

$$s(r,N) \triangleq \sum_{t=r+1}^{N} [e_x(t+\eta_0)e_y(t+\eta_1)e_z(t+\eta_2) - E\{e_x(t+\eta_0)e_y(t+\eta_1)e_z(t+\eta_2)\}] \tag{28}$$

then it holds that

$$E\{|s(r,N)|^2\} \leqslant m_{\max}(N-r) \tag{29}$$

Indeed we examine two cases corresponding to the alternative assumptions (iv)(a) and (iv)(b).
If (iv)(b) holds, then

$$s(r,N) = \sum_{t=r+1}^{N} [e(t+\eta_0')e(t+\eta_1')e(t+\eta_2') - E\{e(t+\eta_0')e(t+\eta_1')e(t+\eta_2')\}]$$

For this sum $m_{\max}\{m_2^3, m_2, m_4, m_6 - m_3^2\}$, where $m_i \triangleq E\{e^i(t)\}$ (see also Reference 7).
If, on the other hand, (iv)(a) holds, then $s(r,N) = \sum_{t=r+1}^{N} e_x(t+\eta_0)e_y y(t+\eta_1)e_z(t+\eta_2)$, which implies

$$\begin{aligned}|s(r,N)|^2 &= \sum_{t=r+1}^{N}\sum_{y=r+1}^{N} E\{e_x(t+\eta_0)e_x(s+\eta_0)\}E\{e_y(t+\eta_1)e_y(s+\eta_1)e_z(t+\eta_2)e_z(s+\eta_2)\} \\ &= \sum_{t=r+1}^{N} E\{e_x(t+\eta_0)^2\}E\{e_y(t+\eta_1)^2 e_z(t+\eta_2)^2\} \\ &= m_{\max}(N-r)\end{aligned}$$

with $m_{\max} \triangleq E\{e_x^2(t)\}\max_{\eta_1-\eta_2} E\{e_y(t+\eta_1)^2 e_z(t+\eta_2)^2\}$.
Using the triangle inequality, we obtain

$$|D_\theta(r,N)| \leqslant \sum_{k=-N}^{N} |\lambda^*(k)| \sum_i |h_{x\theta}(i)| \sum_j |h_{y\theta}(j)| \sum_l |h_{z\theta}(l)|\,|s(r,N)|$$

with $\eta_0 = k-i$, $\eta_1 = -j$ and $\eta_2 = -l$. In view of relations (14d–f) and (29), Lemma 2 yields

$$E\{\sup_\theta |D_\theta(r,N)|^2\} \leqslant C_l^2 C_x^2 C_y^2 C_z^2 M^2 (N-r) \tag{30}$$

Using (30) and condition (R1), Lemma 3 yields

$$\frac{1}{N}\sup_\theta |D_\theta(0,N)| \to 0 \quad \text{w.p.1} \quad \text{as } N\to\infty \tag{31}$$

This completes the proof of Lemma 1.

APPENDIX II

In this appendix we quote as a lemma a useful expression relating multiple modulated projections of cumulants to cumulants of lower order.

Lemma 4

Let $u_i(t) = \sum_\tau g_i(t-\tau)e(\tau)$, $i = 1, \ldots, k$, be a collection of SISO *linear processes* with common driving input the i.i.d. process $e(t)$ and let $G_i(\omega)$ be the corresponding transfer functions. Then for $k > l$

$$\sum_{\tau_1} \cdots \sum_{\tau_{l-l}} \exp\left(-j\sum_{i=1}^{k-l} \omega_i \tau_i\right) \mathrm{cum}\{u_1(\tau_1), \ldots, u_k(\tau_k)\} = \frac{\gamma_{ke}}{\gamma_{le}} G_1(\omega_1) \ldots G_{k-l}(\omega_{k-l}) \mathrm{cum}\{u_{k-l+1}(\tau_{k-l+1}), \ldots, u_k(\tau_k)\}$$

where γ_{ke} ($\gamma_{\lambda e}$) is the kth-order (lth-order) cumulant of $e(t)$) provided that $\sum^{k-l}_{i=1} \omega_i = 0$.

Proof. See Lemma A. 1 of Appendix A in Reference 8.

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