# FRACTIONAL SAMPLING RATE CONVERSION IN THE 3RD ORDER CUMULANT DOMAIN AND APPLICATIONS

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## ABSTRACT

In a variety of problems a random process is observed at different resolutions while knowledge of the corresponding scale conversion ratio usually contains useful information related to problem-specific quantities. A method is proposed which exploits cumulant domain relations of such signals in order to yield fractional estimates of the unknown conversion ratio. The noise insensitivity and shift invariance property of the cumulants offers advantages to the proposed method over signal domain alternatives. These advantages are discussed in two classes of practical problems involving 1-D and 2-D scale converted signals.

## 1. INTRODUCTION

Fractional sampling rate (or scale) conversion of $D$-dimensional processes arises in a variety of signal processing contexts involving signals observed at two different resolutions. The conversion of a signal from an original resolution to a lower one may be due either to the digital signal processing method employed (e.g., deliberate downsampling) or to the sampling mechanism employed for A/D conversion. As fractional numbers are dense in real numbers, fractional resolution conversion ratios can approximate arbitrarily close any resolution ratio. Rate conversion problems arise, for example, in pattern recognition or classification applications where one has to compare incoming signals, acquired at a given (*test*) resolution, to pre-stored data, acquired at a different (usually higher) *reference* resolution. The ratio of these two scales may be unknown. The processing required can be carried out in the signal domain itself; feature extraction or statistics estimation is normally employed, however, in order to move to a domain where comparisons are possible and they can be performed at reduced computational cost.

Third order cumulants are examined as a candidate domain in the present work, as a special case of the more general $k$-th order cumulant domain, addressed in [5]. Motivation for this provide the established *noise robustness* and *phase information retension* properties of the third order cumulant, rendering it an attractive candidate either for feature selection or for statistics estimation. See, e.g., [1] and [8] for definitions, properties and applications of third order cumulants and bispectra. Specifically in this paper relations are established between the third order cumulants (and bispectra) of the signal at the high and the low resolution. These relations can be used either to estimate the third order cumulants of the low resolution signal directly from the cumulants of the high resolution signal or to estimate the ratio of the two resolutions (scales), when unknown.

The obtained relations are used in two representative applications involving sampling rate conversions of 1-D and 2-D stochastic signals respectively.

The first application estimates the velocity of a moving source by exploiting the Doppler effect. The moving object, whose velocity is sought, may either reflect an 1-D narrowband signal (e.g., a tone) transmitted by the detector, or independently emit a generally wideband signal. The signal received at the detector site is a fractionally converted version of the original signal, and the fractional conversion ratio is related to the velocity of the target. Therefore, the target velocity can be obtained via estimation of the scale ratio between the original and the received versions of the signal. Traditional methods rely on the assumption that the measured signals are strictly of narrowband nature (tones). Our approach focuses on the alternative assumption of stochastic signals, possibly measured in low signal to noise ratio conditions.

In the second application, the distance from a textured surface is computed by comparing the resolution of a captured image to that of a reference image. A typical example of such problems is the estimation of the flight altitude of a plane from landscape images taken during flight. Resolution conversion in this case involves two $2 \times 2$ integer, non-singular decimation and expansion matrices.

## 2. THIRD ORDER STATISTICS RELATIONS

Let $y(\mathbf{n})$ denote a rate-converted, discrete variable, $D$-dimensional process obtained from the original process $x(\mathbf{n})$ after fractional rate conversion:

$$y(\mathbf{n}) = \sum_{\mathbf{k}} h(M_{D\times D}\mathbf{n} - L_{D\times D}\mathbf{k})\, x(\mathbf{k}), \qquad (1)$$

where the rate conversion is obtained through a "downsample by matrix $L_{D\times D}$ - filter by $h(\mathbf{n})$ - expand by matrix $M_{D\times D}$" operation and $L_{D\times D}, M_{D\times D}$ are $D \times D$ integer, non-singular, commutative and coprime matrices, [2]. Of practical interest are the cases with $det(M_{D\times D})>det(L_{D\times D})$, meaning that $y(\mathbf{n})$ is observed at a resolution lower than the original (reference) resolution of $x(\mathbf{n})$.

Equation (1) covers the general case where resampling is possibly accompanied by rotation and azimuth changes (see [9], chap. 12). In [5] input - output cumulant expressions are established for the general case of equation (1) with full integer matrices $L_{D\times D}, M_{D\times D}$. For the purposes of the present work, we restrict ourselves to the special case of diagonal matrices $L_{D\times D} = LI_{D\times D}$, $M_{D\times D} = MI_{D\times D}$, where $L, M$ are coprime integers. This corresponds to rescaling by the scalar factor $M/L$ alone, thus preserving the view point from which the signal is observed. This choice is made in order to simplify the mathematical notations, since here the focus is on the use of these relations for the estimation of the resampling ratio in the aforementioned applications. It is straightforward to rewrite the obtained relations, however, for the general case of general diagonal or full matrices $L_{D\times D}, M_{D\times D}$.

Equation (1) is equivalent to a resampling of the continuous signal $x_c(\mathbf{t})$ from which $x(\mathbf{n})$ was originally obtained through sampling, provided that the filter $h(\mathbf{n})$ employed has an ideal lowpass transfer function $H(\omega)$ with gain $L$ and cutoff frequencies $\pi/M$ over all dimensions $d = 1, \ldots, D$.

The *third order cyclic cumulant* of $y(\mathbf{n})$, defined as ([5])

$$\tilde{c}_{3,y}(\mathbf{m}_1, \mathbf{m}_2) \triangleq \frac{1}{L^D} \sum_{\mathbf{n}\in[0\ldots L-1]^D} c_{3,y}(\mathbf{m}_1, \mathbf{m}_2; \mathbf{n}), \qquad (2)$$

where $c_{3,y}(\mathbf{m}_1, \mathbf{m}_2; \mathbf{n})$ is periodic in $(\mathbf{n})$ with period $[0..L-1]^D$, is related to $c_{3,x}(\mathbf{m}_1, \mathbf{m}_2)$ through the following equation:

$$\begin{aligned}\tilde{c}_{3,y}(\mathbf{m}_1, \mathbf{m}_2) &= \frac{1}{L^D} \sum_{\mathbf{s}_1} \sum_{\mathbf{s}_2} c_{3,x}(\mathbf{s}_1, \mathbf{s}_2) \\ &\times\ h_3(M\mathbf{m}_1 - L\mathbf{s}_1, M\mathbf{m}_2 - L\mathbf{s}_2)\ (3)\end{aligned}$$

where $h_3(\mathbf{m}_1, \mathbf{m}_2) \triangleq \sum_{\mathbf{n}} h(\mathbf{n})\, h(\mathbf{n}+\mathbf{m}_1)\, h(\mathbf{n}+\mathbf{m}_2)$ is the triple correlation of the decimation $D$-dimensional filter $h(\mathbf{n})$. See [3], [6] for definitions and properties of cyclic moments and cumulants.

Equation (3) can be transformed to the frequency domain, to yield relations between the cyclic bispectrum of the low resolution signals and the bispectrum of the reference resolution signals, by exploiting the coprimeness of $L, M$:

$$\begin{aligned}\tilde{C}_{3,y}(\omega_1, \omega_2) &= \frac{1}{M^D} \sum \sum\nolimits_{\mathbf{l}_1,\mathbf{l}_2\in[0\ldots M-1]^D} \\ &\times\ C_{3,x}\left(\frac{L\omega_1 + 2\pi L\mathbf{l}_1}{M}, \frac{L\omega_2 + 2\pi L\mathbf{l}_2}{M}\right) \\ &\times\ H_3\left(\frac{\omega_1 + 2\pi\mathbf{l}_1}{M}, \frac{\omega_2 + 2\pi\mathbf{l}_2}{M}\right) \qquad (4)\end{aligned}$$

The RHS of equation (4) is a summation of frequency shifted replicas of $C_{3,x}(\omega_1, \omega_2)$, each replica shrinked by the scaling factor $M/L$. It is interesting to notice the resemblance that equation (4) bears to the corresponding input - output relation between the Fourier transforms of deterministic signals.

## 3. ESTIMATION OF THE RESOLUTION CONVERSION RATIO

The input - output relations given in the previous section allow for the computation of the resolution conversion ratio, $L/M$, provided that both $x(\mathbf{n})$ and $y(\mathbf{n})$ are available and the blurring mechanism $h(\mathbf{n})$ is known. The method proposed in the sequel for the computation of $L/M$ relies on *matching the cumulants of the measured signal* $y(\mathbf{n})$ *to successive resolution-converted versions of the cumulants of the original signal* $x(\mathbf{n})$. Although computationally demanding, this method is shown to converge to the true resolution conversion ratio.

Before proceeding to the description of the proposed method, it should be emphasized that addressing this problem in the third order statistics rather than the signal domain offers two advantages.

1. Statistical similarity is exploited, which means that valid results are obtained even if $y(\mathbf{n})$ and $x(\mathbf{n})$ correspond to different realizations of a stochastic process. This feature is of great importance in situations where the reference (high resolution) and the test (low resolution) signals are not acquired simultaneously. This is the case, e.g., with pattern classification applications that use pre-stored data.
2. The well known immunity of the third order statistics to a wide class of additive noises, makes the proposed method appropriate for situations where only noisy data is available.

*Proposed Method* :

**step 1** : Estimate the cumulant of the reference signal $x(\mathbf{n})$, $\hat{c}_{3x}(\mathbf{m}_1,\mathbf{m}_2)$, and the cyclic cumulant of the test signal $y(\mathbf{n})$, $\hat{\hat{c}}_{3y}(\mathbf{m}_1,\mathbf{m}_2)$. The asymptotically consistent estimator of the cyclic cumulant proposed in [3] can be employed. This estimator in the present set up takes the form

$$\hat{\hat{c}}_{3y}(\mathbf{m}_1,\mathbf{m}_2)=\frac{1}{T_y}\sum_{\mathbf{t}\in\mathbf{W}_y} y(\mathbf{t})y(\mathbf{t}+\mathbf{m}_1)y(\mathbf{t}+\mathbf{m}_2), \tag{5}$$

where $\mathbf{W}_y$ is the set of all available data samples of signal $y(\mathbf{n})$, and $T_y$ is the cardinal number of $\mathbf{W}_y$. Note that in practice the conventional estimator $\hat{c}_{3x}(\mathbf{m}_1,\mathbf{m}_2)$ for the (non-cyclic) cumulant of $x(\mathbf{n})$ is implemented in the same way.

**step 2** : Define a partition $\{q_n\}, n=1,2,\ldots,N$ of the scale (resolution) interval $(0,1]$ with $q_n$ chosen as fractional numbers $L_n/M_n$, where $L_n \leq M_n$ and $(L_n,M_n)$ are coprime integers.

**step 3** : For $n=1,2,\ldots,N$,

1. Compute the triple correlation $h_3^{(n)}(\mathbf{m}_1,\mathbf{m}_2)$ of the resampling filter $h(\mathbf{n})^{(n)}$, which should have gain $L_n$ and cutoff frequency $\pi/M_n$.
2. Estimate the cyclic cumulant $\hat{\hat{c}}_{3y}^{(n)}(\mathbf{m}_1,\mathbf{m}_2)$ of $y^{(n)}(\mathbf{n})$, which is a resampled version of the reference signal $x(\mathbf{n})$ at a sampling rate $M_n/L_n$ times lower than that of $x(\mathbf{n})$.

   This estimator can be implemented as in equation (3), using $L_n, M_n, h_3^{(n)}$ in place of $L, M, h_3$.
3. Compute the similarity index

$$f(\frac{L_n}{M_n})\triangleq\sum_{\mathbf{m}_1,\mathbf{m}_2}|\hat{\hat{c}}_{3,y}(\mathbf{m}_1,\mathbf{m}_2)-\hat{\hat{c}}_{3,y}^{(n)}(\mathbf{m}_1,\mathbf{m}_2)|^2. \tag{6}$$

**step 4** : Obtain an estimate of the conversion ration $L/M$ as the point $q_n=L_n/M_n$ of the global minimum of index $f(q_n)$ over all $n=1,2,\ldots,N$. The estimate of the conversion ration $L/M$ can be drown arbitrarily close to the true ratio $L/M$ by repeatedly refining the partition $q_n$ of $(0,1]$. Local refinement can be used, in a process of zooming into the neighborhood of the initial the global minimum.

Comment:

The similarity index $f(q_n)$ is in general a nonconvex function of $q_n$. However, it has been observed that $f(q_n)$ exhibits a deep global minimum at $L/M$. Also in the neighborhood of the global minimum it assumes a convex form. Therefore, in the neighborhood of the global minimum the refinement process can be driven by fast minimization algorithms such as the Fibonacci and Golden Section methods, ([7]).

## 4. APPLICATIONS

Use of the proposed method for the estimation of the unknown sampling rate conversion ratio $L/M$ is investigated in the sequel, in an 1-D and a 2-D problem.

### 4.1. Velocity estimation via Doppler effect

A signal $x(t)$ reflected by an object moving with velocity $v$ is observed at the source (detector) site as $y(t)=x(\alpha t)$, where $\alpha=\frac{c+v}{c-v}$ and $c$ is the velocity of transmitted signal. This corresponds to a rescaling of the transmitted signal $x(t)$ by a factor $\alpha$. A similar relation with $\alpha=\frac{c+v}{c}$ holds in the case that the moving object itself is emitting the signal $x(n)$, rather than reflecting it.

Conventional methods assume that $x(t)$ is a narrowband signal, usually a sinusoid, and measure the frequency shift between the transmitted and the received signals as a means to compute $\alpha$ and then $v$. This narrowband assumption is not always met, either because of physical constraints of the emitters, or when transmission of simusoidal signals is to be avoided for security reasons.

In such cases the method proposed here can be used, approximating $\alpha$ arbitrarily close by a fraction $M/L$. The proposed method offers the advantages of (i) being immune to a class of additive noises, as a result of its third order cumulant basis, and (ii) allowing the widespreading of the frequency contents of the transmitted signal, necessary under certain applications.

### 4.2. Relative distances from textured images

Pictures of a textured surface, acquired from varying distances, represent versions of the surface under different space scalings. The scale ratio between two such pictures (a reference and a test one) can be used to obtain the distance of the camera from the surface, for the test picture. This problem arises in various image processing tasks, as for example: (i) when landscape images are taken from different altitudes, which are unknown at the processing time, (ii) in motion information extraction from video sequences depicting objects that move towards/off the camera, (iii) in medical imaging application where tissue images are processed, etc.

If $I(\mathbf{t})$ is the analog image and $x(\mathbf{t})$ the analog form of $I(\mathbf{t})$ observed from distance $d_x$, then $x(\mathbf{t}) = I(\frac{d_x}{f}\mathbf{t})$, where $f$ is the focal distance of the camera. Assuming that the test picture $y(\mathbf{t})$ is taken from an unknown distance $d_y$, then $y(\mathbf{t}) = I(\frac{d_y}{f}\mathbf{t}) = x(\frac{d_y}{d_x}\mathbf{t})$. The proposed method can be used to approximate arbitrarily close the ratio $d_y/d_x$ by a fractional number. Therefore, if $d_x$ is known, $d_y$ can be computed.

Comments:

Scale registration methods applied directly to the image rather than to the cumulant domain can be also used along the lines of the proposed method. The advantages of using cumulant statistics, though, are:

1. The fact that statistical similarity is exploited allows to obtain valid distance estimates even when the test and the reference images do not depict the same surface, provided that they correspond to images possessing similar statistical structure.

2. The shift invariance of the cumulant domain allows for the comparison of images that are not necessarily aligned in space.

Possible generalization of the algorithm to address problems involving general diagonal or non-diagonal decimation and interpolation matrices $L, M$ will make the comparisons insensitive to azimuth / rotation changes.

## 5. CONCLUSIONS

A cumulant based method for the estimation of fractional scale conversion ratio of signals is proposed in the present work. Expressions relating the higher-order statistics of signals observed at two different resolutions (scales) are quoted and used in a practical algorithm for scale conversion ratio estimation. Application of the proposed algorithm in problems involving 1-D and 2-D signals are outlined, and relative merits of the proposed approach due to the use of higher order statistics are discussed.

## 6. REFERENCES


[1] D.R.Brillinger, "Time series: Data Analysis and Theory," *McGraw-Hill,* 1981.

[2] T.Chen, P.P.Vaidyanathan, "Commutativity of D-Dimensional Decimation and Expansion Matrices, and Application to Rational Decimation Systems," *Proc. ICASSP'92,* vol. IV, pp. 637-640, San Francisco, USA, March 1992.

[3] A.V.Dandawate, G.B.Giannakis, "Asymptotic theory of mixed time averages and kth-order cyclic-moment and cumulant statistics," *IEEE Trans. on Information Theory,* vol. 41, pp. 216-232, January 1995.

[4] A.Delopoulos, S.Kollias, "Optimal Filterbanks for Signal Reconstruction from Noisy Subband Components," *IEEE Trans. on Signal Proc.,* vol. 44, no. 2, pp. 212-224, Feb. 1996.

[5] A.Delopoulos, M.Rangoussi, "Cumulant relations of multidimensional processes observed at rationally related resolutions," *Proc. Intl. Workshop on Sampling Theory and Applications,* Aveiro, Portugal, June 1997 (to be presented).

[6] G.B.Giannakis, G.Zhou, "Parameter estimation of cyclostationary amplitude modulated time series with application to missing observations," *IEEE Trans. on Signal Processing,* vol. 42, no. 9, pp. 2408-2419, Sept. 1994.

[7] G.Karmanov, "Mathematical Programming," *Mir Publishers,* Moscow, 1987.

[8] J.Mendel, "Tutorial on Higher-Order Statistics (Spectra) in Signal Processing and System Theory: Theoretical Results and some Applications," *Proc. of the IEEE,* vol. 79, no. 3, pp. 278-305, March 1991.

[9] P.P.Vaidyanathan, "Multirate Systems and Filter banks," *Prentice Hall,* Englewood Cliffs, New Jersey, 1993.